

STABILITY OF THE CHARI-LOKTEV BASES FOR LOCAL WEYL MODULES OF $\mathfrak{sl}_{r+1}[t]$

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ABSTRACT. We prove stability of the Chari-Loktev bases with respect to the inclusions of local Weyl modules of the current algebra $\mathfrak{sl}_{r+1}[t]$. This is conjectured in [8] and proved the $r = 1$ case in [7]. Local Weyl modules being known to be Demazure submodules in the level one representations of the affine Lie algebra $\widehat{\mathfrak{sl}_{r+1}}$, we obtain, by passage to the direct limit, bases for the level one representations themselves.

1. INTRODUCTION

Let \mathfrak{g} be a finite-dimensional complex simple Lie algebra and $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ be its current algebra. Local Weyl modules, introduced by Chari and Pressley [3] are interesting finite-dimensional $\mathfrak{g}[t]$ -modules. These modules are characterized by the following universal property: any finite-dimensional $\mathfrak{g}[t]$ -module generated by a one-dimensional highest weight space, is a quotient of a local Weyl module. Corresponding to every dominant integral weight λ of \mathfrak{g} , there is one local Weyl module denoted by $W(\lambda)$.

In [3], for $\mathfrak{g} = \mathfrak{sl}_2$, Chari and Pressley also produced monomial bases for local Weyl modules. Later Chari and Loktev [2] extended the construction of these bases to $\mathfrak{g} = \mathfrak{sl}_{r+1}$. Using this bases they also showed that the local Weyl modules are $\mathfrak{g}[t]$ -stable Demazure modules occurring in a level one representations of the affine Lie algebra $\widehat{\mathfrak{g}}$. As a consequence, we get an embedding of local Weyl modules $W(\lambda) \hookrightarrow W(\lambda + k\theta)$, where θ is the long root and k is a non-negative integer. It is important to note that for every non-negative integer k , the local Weyl module $W(\lambda + k\theta)$ can be realized as $\mathfrak{g}[t]$ -stable Demazure module occurring in a fixed level one representation of $\widehat{\mathfrak{g}}$; we shall denote this level one representation here by V . Thus we have a chain of inclusions:

$$W(\lambda) \hookrightarrow W(\lambda + \theta) \hookrightarrow \cdots \hookrightarrow W(\lambda + k\theta) \hookrightarrow W(\lambda + (k+1)\theta) \hookrightarrow \cdots (\hookrightarrow V), \quad (1.1)$$

such that the union of the modules in the chain equals V .

For $\mathfrak{g} = \mathfrak{sl}_2$, it is shown in [7] that after a suitable normalization, the Chari-Pressley bases behave well with respect to the inclusions in (1.1). More over in the limit, these bases stabilize and give a nice monomial basis for V . For $\mathfrak{g} = \mathfrak{sl}_{r+1}$, we consider the Chari-Loktev (CL) bases for local Weyl modules. In [8], an elegant combinatorial description for its parameterizing set is given: namely, as the set of *partition overlaid patterns* (POPs). More over a weight preserving injective map between the parameterizing sets of the bases for $W(\lambda + k\theta)$ and $W(\lambda + (k+1)\theta)$ is given, and also conjectured

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that after a suitable normalization the CL bases also have the stability property with respect to the inclusions in (1.1). The purpose of this paper is to prove this conjecture.

More precisely, let \mathbb{P}_λ denote the parametrizing set of the CL basis for $W(\lambda)$: the elements of \mathbb{P}_λ are POPs with bounding sequence $\underline{\lambda}$, where $\underline{\lambda}$ is an integer tuple corresponding to λ . In [8], for each non-negative integer k , a weight preserving embedding from $\mathbb{P}_{\lambda+k\theta}$ into $\mathbb{P}_{\lambda+(k+1)\theta}$ is given. Thus we have a chain $\mathbb{P}_\lambda \hookrightarrow \mathbb{P}_{\lambda+\theta} \hookrightarrow \mathbb{P}_{\lambda+2\theta} \hookrightarrow \dots$. Given an element \mathfrak{P} of \mathbb{P}_λ and a non-negative integer k , let \mathfrak{P}^k be its image in $\mathbb{P}_{\lambda+k\theta}$, and let $v_{\mathfrak{P}^k}$ be the corresponding normalized CL basis element. Consider the sequence $v_{\mathfrak{P}^k}, k = 0, 1, 2, \dots$, of elements in V . We prove that this sequence stabilizes for large k (see Theorem 3.4). Passing to the direct limit, we obtain a basis for V consisting of the stable CL basis elements (see §3.3).

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2. NOTATION AND PRELIMINARIES

Throughout the paper, \mathbb{C} denotes the field of complex numbers, \mathbb{Z} the set of integers, \mathbb{N} the set of positive integers, $\mathbb{Z}_{\geq 0}$ the set of non-negative integers, $\mathbb{C}[t]$ the polynomial ring, $\mathbb{C}[t, t^{-1}]$ the ring of Laurent polynomials, and $\mathbf{U}(\mathfrak{a})$ the universal enveloping algebra corresponding to a complex Lie algebra \mathfrak{a} .

2.1. The Lie algebra \mathfrak{sl}_{r+1} . Let $\mathfrak{g} = \mathfrak{sl}_{r+1}$, the Lie algebra of $(r+1) \times (r+1)$ trace zero matrices over the field \mathbb{C} of complex numbers. Let \mathfrak{h} be the *standard* Cartan subalgebra of \mathfrak{g} consisting of trace zero diagonal matrices. Let \mathfrak{b} be the *standard* Borel subalgebra of \mathfrak{g} consisting of upper triangular matrices. For $1 \leq i \leq r+1$, let $\varepsilon_i \in \mathfrak{h}^*$ be the projection to the i^{th} co-ordinate. Set $I = \{1, 2, \dots, r\}$. Let $\varpi_i = \varepsilon_1 + \dots + \varepsilon_i, i \in I$, be the set of fundamental weights of \mathfrak{g} . Let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}, i \in I$, be a set of simple roots of \mathfrak{g} , and $\alpha_{i,j} = \alpha_i + \dots + \alpha_j, 1 \leq i \leq j \leq r$, be the set positive roots of \mathfrak{g} with respect to \mathfrak{b} . Let $\theta = \alpha_{1,r}$ be the highest root of \mathfrak{g} . For $1 \leq i, j \leq r+1$, let $E_{i,j}$ be the $(r+1) \times (r+1)$ matrix with 1 in the $(i, j)^{\text{th}}$ position and 0 elsewhere. Define subalgebras \mathfrak{n}^\pm of \mathfrak{g} by

$$\mathfrak{n}^\pm = \bigoplus_{1 \leq i \leq j \leq r} \mathbb{C} x_{i,j}^\pm,$$

where $x_{i,j}^+ = E_{i,j+1}$ and $x_{i,j}^- = E_{j+1,i}$. Now we have the following decomposition: $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. For $x, y \in \mathfrak{g}$, let $(x|y) := \text{trace}(xy)$ be the normalized invariant bilinear form on \mathfrak{g} .

The weight lattice P , the set P^+ of dominant integral weights, and the root lattice Q of \mathfrak{g} are defined as follows:

$$P = \sum_{i \in I} \mathbb{Z} \varpi_i, \quad P^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \varpi_i, \quad \text{and} \quad Q = \sum_{i \in I} \mathbb{Z} \alpha_i.$$

For $\lambda = m_1 \varpi_1 + \dots + m_r \varpi_r \in P^+$, we associate an integer tuple $\underline{\lambda} = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 0)$, where $\lambda_i := m_i + \dots + m_r$. Given an integer tuple $\underline{\lambda} = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 0)$, we associate an element λ of P^+ by $\lambda = \lambda_1 \varepsilon_1 + \dots + \lambda_r \varepsilon_r$.

2.2. The affine Lie algebra $\widehat{\mathfrak{g}}$. Let $\widehat{\mathfrak{g}}$ be the (untwisted) affine Lie algebra corresponding to \mathfrak{g} defined by

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where c is central and the other Lie brackets are given by

$$\begin{aligned} [x \otimes t^m, y \otimes t^n] &= [x, y] \otimes t^{m+n} + m\delta_{m,-n}(x|y)c, \\ [d, x \otimes t^m] &= m(x \otimes t^m), \end{aligned}$$

for all $x, y \in \mathfrak{g}$ and integers m, n . The Lie subalgebras $\widehat{\mathfrak{h}}$ and $\widehat{\mathfrak{b}}$ of $\widehat{\mathfrak{g}}$ are given by

$$\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d, \quad \widehat{\mathfrak{b}} = \mathfrak{g} \otimes t\mathbb{C}[t] \oplus \mathfrak{b} \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

We regard \mathfrak{h}^* as a subspace of $\widehat{\mathfrak{h}}^*$ by setting $\langle \lambda, c \rangle = \langle \lambda, d \rangle = 0$ for $\lambda \in \mathfrak{h}^*$. Let $\delta, \Lambda_0 \in \widehat{\mathfrak{h}}^*$ be given by

$$\langle \delta, \mathfrak{h} + \mathbb{C}c \rangle = 0, \quad \langle \delta, d \rangle = 1, \quad \langle \Lambda_0, \mathfrak{h} + \mathbb{C}d \rangle = 0, \quad \langle \Lambda_0, c \rangle = 1.$$

There is a non-degenerate, symmetric, \widehat{W} -invariant, bilinear form $(\cdot|\cdot)$ on $\widehat{\mathfrak{h}}^*$, given by requiring that \mathfrak{h}^* be orthogonal to $\mathbb{C}d + \mathbb{C}\Lambda_0$, together with the relations

$$(\alpha_i|\alpha_i) = 2, \quad (\alpha_i|\alpha_j) = -\delta_{j,i+1}, \quad \forall 1 \leq i < j \leq r, \quad (\delta|\delta) = (\Lambda_0|\Lambda_0) = 0, \quad \text{and} \quad (\delta|\Lambda_0) = 1.$$

The elements $\alpha_0 = \delta - \theta, \alpha_1, \dots, \alpha_r$ are the simple roots of $\widehat{\mathfrak{g}}$ and the corresponding coroots are $\alpha_0^\vee = c - \theta^\vee, \alpha_1^\vee, \dots, \alpha_r^\vee$. Set $\widehat{I} = I \cup \{0\}$. Let e_i, f_i ($i \in \widehat{I}$) denote the Chevalley generators of $\widehat{\mathfrak{g}}$; these are given by

$$e_0 = x_{1,r}^- \otimes t, \quad f_0 = x_{1,r}^+ \otimes t^{-1}, \quad e_i = x_{i,i}^+, \quad f_i = x_{i,i}^-, \quad \forall i \in I.$$

For $\alpha \in R, s \in \mathbb{Z}$, set

$$x_\alpha := \begin{cases} x_{i,j}^+, & \alpha = \alpha_{i,j} \in R^+, \\ x_{i,j}^-, & \alpha = -\alpha_{i,j} \in R^-, \end{cases} \quad \text{and} \quad x_{\alpha+s\delta} := x_\alpha \otimes t^s. \quad (2.1)$$

The weight lattice (resp. the set of dominant integral weights) is defined by

$$\widehat{P} \text{ (resp. } \widehat{P}^+) = \{\Lambda \in \widehat{\mathfrak{h}}^* : \langle \Lambda, \alpha_p^\vee \rangle \in \mathbb{Z} \text{ (resp. } \mathbb{Z}_{\geq 0}), \forall p \in \widehat{I}\}.$$

For an element $\Lambda \in \widehat{P}$, the integer $\langle \Lambda, c \rangle$ is called the *level* of Λ .

2.3. The Weyl group of $\widehat{\mathfrak{g}}$. For each $p \in \widehat{I}$, the fundamental reflection s_{α_p} (or s_p) is given by

$$s_p(\Lambda) = \Lambda - \langle \Lambda, \alpha_p^\vee \rangle \alpha_p, \quad \forall \Lambda \in \widehat{\mathfrak{h}}^*.$$

The subgroup \widehat{W} of $GL(\widehat{\mathfrak{h}}^*)$ generated by all fundamental reflections $s_p, p \in \widehat{I}$ is called the affine Weyl group. We regard W naturally as a subgroup of \widehat{W} . Given $\alpha \in \mathfrak{h}^*$, let $t_\alpha \in GL(\widehat{\mathfrak{h}}^*)$ be defined by

$$t_\alpha(\Lambda) = \Lambda + (\Lambda|\delta)\alpha - (\Lambda|\alpha)\delta - \frac{1}{2}(\Lambda|\delta)(\alpha|\alpha)\delta, \quad \text{for } \Lambda \in \widehat{\mathfrak{h}}^*.$$

It is easy to see that

$$t_\alpha t_\beta = t_{\alpha+\beta} \quad \text{and} \quad wt_\alpha w^{-1} = t_{w\alpha}, \quad \forall \alpha, \beta \in \mathfrak{h}^*, w \in W. \quad (2.2)$$

The translation subgroup T_Q of \widehat{W} is defined by $T_Q := \{t_\alpha \in GL(\widehat{\mathfrak{h}}^*) : \alpha \in Q\}$.

The following proposition gives the relation between W and \widehat{W} .

Proposition 2.1. [6, Proposition 6.5] $\widehat{W} = W \ltimes T_Q$.

The extended affine Weyl group \widetilde{W} is the semi-direct product

$$\widetilde{W} := W \ltimes T_P,$$

where $T_P = \{t_\alpha \in GL(\widehat{\mathfrak{h}}^*) : \alpha \in P\}$. For $i \in I$, consider the element $\sigma_i = t_{\varpi_i} w_{0,i} w_0 \in \widetilde{W}$. It is an automorphism of the Dynkin diagram of $\widehat{\mathfrak{g}}$;

$$\sigma_i \alpha_p = \alpha_{i+p \pmod{r+1}}, \quad \forall p \in \widehat{I}, \quad \text{and } \sigma \rho = \rho.$$

Here, $\rho \in \widehat{\mathfrak{h}}^*$ is the Weyl vector, defined by $\langle \rho, \alpha_p^\vee \rangle = 1, \forall p \in \widehat{I}$, and $\langle \rho, d \rangle = 0$. We also have $\widetilde{W} = \widehat{W} \ltimes \Sigma$, where Σ is the subgroup generated by $\{\sigma_i : i \in I\}$ (see [1, Chapter VI]).

2.4. Irreducible modules of $\widehat{\mathfrak{g}}$. Given $\Lambda \in \widehat{P}^+$, let $L(\Lambda)$ be the irreducible $\widehat{\mathfrak{g}}$ -module with highest weight Λ . It is the cyclic $\widehat{\mathfrak{g}}$ -module generated by v_Λ , with defining relations:

$$h v_\Lambda = \langle \Lambda, h \rangle v_\Lambda, \quad \forall h \in \widehat{\mathfrak{h}}, \quad (2.3)$$

$$e_p v_\Lambda = 0, \quad \forall p \in \widehat{I}, \quad (2.4)$$

$$f_p^{\langle \Lambda, \alpha_p^\vee \rangle + 1} v_\Lambda = 0, \quad \forall p \in \widehat{I}. \quad (2.5)$$

It has weight space decomposition $L(\Lambda) = \bigoplus_{\mu \in \widehat{\mathfrak{h}}^*} L(\Lambda)_\mu$. The μ for which $L(\Lambda)_\mu \neq 0$ are the *weights* of $L(\Lambda)$.

The following two results are well-known:

Proposition 2.2. [6] *Let $\Lambda \in \widehat{P}^+$ is of level 1. Then*

- (1) *the set of weights of $L(\Lambda)$ is $\{t_\alpha(\Lambda) - m\delta \mid \alpha \in Q, m \in \mathbb{Z}_{\geq 0}\}$,*
- (2) *for $\alpha \in Q$ and $m \in \mathbb{Z}_{\geq 0}$, we have*

$$\dim L(\Lambda)_{t_\alpha(\Lambda) - m\delta} = \text{the number of } r\text{-colored partitions of } m \text{ (see §2.9)}.$$

Theorem 2.3. [5] *Given $m \in \mathbb{Z}_{\geq 0}$, every element of the weight space of $L(\Lambda_0)$ of weight $\Lambda_0 - m\delta$ can be written as $g_m v_{\Lambda_0}$ for some polynomial g_m in $\alpha_i^\vee t^{-j}, i \in I, j \in \mathbb{N}$.*

We let $\Lambda_i := \sigma_i \Lambda_0$ for $i \in I$. Then, $\Lambda_0, \Lambda_1, \dots, \Lambda_r$ are (a choice of) fundamental weights corresponding to the coroots $\alpha_0^\vee, \alpha_1^\vee, \dots, \alpha_r^\vee$, i.e., $\langle \Lambda_p, \alpha_q^\vee \rangle = \delta_{p,q}$ for $p, q \in \widehat{I}$. We let v_{Λ_p} denote a highest weight vector of $L(\Lambda_p)$ for $p \in \widehat{I}$.

2.5. The current algebra and its Weyl modules. The current algebra $\mathfrak{g}[t] := \mathfrak{g} \otimes \mathbb{C}[t]$ is a Lie algebra with Lie bracket is obtained from that of \mathfrak{g} by extension of scalars to $\mathbb{C}[t]$:

$$[x \otimes t^m, y \otimes t^n] := [x, y] \otimes t^{m+n}, \quad \forall x, y \in \mathfrak{g}, m, n \in \mathbb{Z}_{\geq 0}.$$

Definition 2.4. (see [2, §1.2.1]) Given $\lambda \in P^+$, the *local Weyl module* $W(\lambda)$ is the cyclic $\mathfrak{g}[t]$ -module with generator w_λ and relations:

$$(\mathfrak{n}^+ \otimes t\mathbb{C}[t]) w_\lambda = 0, \quad (h \otimes t^s) w_\lambda = \langle \lambda, h \rangle \delta_{s,0}, \quad \forall h \in \mathfrak{h}, s \in \mathbb{Z}_{\geq 0}, \quad f_i^{\langle \lambda, \alpha_i^\vee \rangle + 1} w_\lambda = 0, \quad \forall i \in I.$$

2.6. Weyl modules as Demazure modules. Given $w \in \widehat{W}$ and $\Lambda \in \widehat{P}^+$, define a $\widehat{\mathfrak{b}}$ -submodule $V_w(\Lambda)$ of $L(\Lambda)$ by

$$V_w(\Lambda) := \mathbf{U}(\widehat{\mathfrak{b}}) L(\Lambda)_{w\Lambda}.$$

We call the $\widehat{\mathfrak{b}}$ -module $V_w(\Lambda)$ as the *Demazure module* of $L(\Lambda)$ associated to w . More generally, given an element w of the extended affine Weyl group \widehat{W} , we write $w = u\tau$ with $u \in \widehat{W}$, $\tau \in \Sigma$ and define, following [4], the associated Demazure module by $V_w(\Lambda) := V_u(\tau(\Lambda))$.

The following theorem identifies the $\mathfrak{g}[t]$ -stable Demazure modules with the local Weyl modules.

Theorem 2.5. [2, 4] *Given $\lambda \in P^+$, the local Weyl module $W(\lambda)$ is isomorphic to the $\mathfrak{g}[t]$ -stable Demazure module $V_{t_{w_0\lambda}}(\Lambda_0)$, as modules for the current algebra $\mathfrak{g}[t]$.*

2.7. Inclusions of Weyl modules. Let $\lambda \in P^+$. Let $\underline{\lambda} : \lambda_1 \geq \dots \geq \lambda_r \geq \lambda_{r+1} = 0$ be the corresponding tuple. Let $i_\lambda \in I \cup \{0\}$ be the remainder when $\sum_{i=1}^{r+1} \lambda_i$ is divided by $r+1$. Set ϖ_0 as the zero element in \mathfrak{h}^* . It is easy to see that $\lambda - \varpi_{i_\lambda} \in Q$. Since $w_0\Lambda_0 = \Lambda_0 = w_0w_{0,i_\lambda}\Lambda_0$, using (2.2), we have

$$t_{w_0\lambda}\Lambda_0 = t_{w_0(\lambda - \varpi_{i_\lambda})}t_{w_0\varpi_{i_\lambda}}(\Lambda_0) = t_{w_0(\lambda - \varpi_{i_\lambda})}w_0\sigma_{i_\lambda}(\Lambda_0). \quad (2.6)$$

Thus from Theorem 2.5, we get

$$W(\lambda) \cong_{\mathfrak{g}[t]} V_{t_{w_0(\lambda - \varpi_{i_\lambda})}w_0}(\Lambda_{i_\lambda}) \subset L(\Lambda_{i_\lambda}). \quad (2.7)$$

For every $k \geq 0$, it is important to note that $i_{\lambda+k\theta} = i_\lambda$ and hence $W(\lambda + k\theta)$ is also a Demazure submodule of $L(\Lambda_{i_\lambda})$.

For every $w \in W$, it is well-known that

$$t_{w_0(\lambda - \varpi_{i_\lambda})}w \leq t_{w_0(\lambda + \theta - \varpi_{i_\lambda})}w \leq \dots \leq t_{w_0(\lambda + k\theta - \varpi_{i_\lambda})}w \leq t_{w_0(\lambda + (k+1)\theta - \varpi_{i_\lambda})}w \leq \dots,$$

where \leq is the Bruhat order on the affine Weyl group. Hence using (2.7), we get a chain of Demazure submodules of $L(\Lambda_{i_\lambda})$:

$$W(\lambda) \hookrightarrow W(\lambda + \theta) \hookrightarrow \dots \hookrightarrow W(\lambda + k\theta) \hookrightarrow W(\lambda + (k+1)\theta) \hookrightarrow \dots \quad (\hookrightarrow L(\Lambda_{i_\lambda})), \quad (2.8)$$

such that union of the modules in the chain equals $L(\Lambda_{i_\lambda})$.

2.8. Partitions. A *partition* is a non-increasing sequence of non-negative integers that is eventually zero. The non-zero elements of the sequence are called the *parts* of the partition. If the sum of the parts of a partition $\underline{\pi} : \pi_1 \geq \pi_2 \geq \dots$ is m , then the partition is said to be a *partition of m* , and we write $|\underline{\pi}| = m$.

2.8.1. Partition fits into a rectangle. Let d, d' are non-negative integers. We say that a partition *fits into a rectangle (d, d')* , if the number of parts is at most d and every part is at most d' .

2.9. Colored partitions. Let r be a positive integer. An *r -colored partition* is a partition in which each part is assigned an integer between 1 and r . The number assigned to a part is its color. We may think of an r -colored partition as an ordered r -tuple $(\underline{\pi}^1, \dots, \underline{\pi}^r)$ of partitions: the partition $\underline{\pi}^i$ consists of all parts of color i of the r -colored partition. An r -colored partition of a non-negative integer m is an r -colored partition with $|\underline{\pi}^1| + \dots + |\underline{\pi}^r| = m$.

2.10. Gelfand-Tsetlin patterns. A *Gelfand-Tsetlin (GT) pattern* (or just *pattern*) \mathcal{P} is an array of integral row vectors $\underline{\lambda}^1, \dots, \underline{\lambda}^r, \underline{\lambda}^{r+1}$:

$$\begin{array}{ccccccc} & & & & \lambda_1^1 & & \\ & & & & \lambda_1^2 & & \lambda_2^2 \\ & & & \dots & \dots & & \dots \\ & & \lambda_1^r & & \dots & & \lambda_r^r \\ \lambda_1^{r+1} & & \lambda_2^{r+1} & & \dots & & \lambda_{r+1}^{r+1} \end{array}$$

subject to the following conditions:

$$\lambda_i^{j+1} \geq \lambda_i^j \geq \lambda_{i+1}^{j+1}, \quad \forall 1 \leq i \leq j \leq r.$$

We call the last sequence $\underline{\lambda}^{r+1}$ of the pattern \mathcal{P} its *bounding sequence*.

Fix $\lambda \in P^+$ and a pattern $\mathcal{P} : \underline{\lambda}^1, \dots, \underline{\lambda}^r, \underline{\lambda}^{r+1}$ with bounding sequence $\underline{\lambda}$.

2.10.1. The weight of a pattern. The *weight* $\text{wt } \mathcal{P} \in \mathfrak{h}^*$ of \mathcal{P} is defined by

$$\text{wt } \mathcal{P} := a_1 \varepsilon_1 + a_2 \varepsilon_2 + \dots + a_{r+1} \varepsilon_{r+1}, \quad \text{where } a_j = \sum_{i=1}^j \lambda_i^j - \sum_{i=1}^{j-1} \lambda_i^{j-1}.$$

Note that $a_1 + a_2 + \dots + a_{r+1} = \lambda_1 + \lambda_2 + \dots + \lambda_r$.

2.10.2. Differences of a pattern. For $1 \leq i \leq j \leq r$, the *differences* $d_{i,j}(\mathcal{P})$ and $d'_{i,j}(\mathcal{P})$ (or just $d_{i,j}$ and $d'_{i,j}$ if \mathcal{P} is clear from the context) of \mathcal{P} are given by

$$d_{i,j}(\mathcal{P}) := \lambda_i^{j+1} - \lambda_i^j \quad \text{and} \quad d'_{i,j}(\mathcal{P}) := \lambda_i^j - \lambda_{i+1}^{j+1}.$$

2.10.3. Area of a pattern. The *triangular area* or just *area* $\triangle(\mathcal{P})$ of \mathcal{P} is defined by

$$\triangle(\mathcal{P}) := \sum_{1 \leq i \leq j \leq r} d_{i,j} d'_{i,j}.$$

2.10.4. Trapezoidal area. The *trapezoidal area* $\square(\mathcal{P})$ of a pattern \mathcal{P} is defined by

$$\square(\mathcal{P}) := \sum_{1 \leq i \leq j \leq r} d_{i,j} \left(\sum_{p=i}^j d'_{p,j} \right).$$

2.10.5. Shift of a pattern. For $k \in \mathbb{Z}_{\geq 0}$, the *shift* \mathcal{P}^k of \mathcal{P} by k is a pattern with bounding sequence $\underline{\lambda} + k\underline{\theta}$: suppose that $\underline{\eta}^1, \dots, \underline{\eta}^r, \underline{\eta}^{r+1}$ be the rows of \mathcal{P}^k , then

$$\eta_i^j := \begin{cases} \lambda_i^j + 2k, & i = 1 \text{ and } 1 < j \leq r+1, \\ \lambda_i^j, & 1 < i = j \leq r+1, \\ \lambda_i^j + k, & \text{otherwise.} \end{cases}$$

We observe for $k \in \mathbb{Z}_{\geq 0}$ that

$$d_{i,j}(\mathcal{P}^k) = d_{i,j}(\mathcal{P}) + \delta_{i,j}k, \quad d'_{i,j}(\mathcal{P}^k) = d'_{i,j}(\mathcal{P}) + \delta_{1,i}k, \quad \forall 1 \leq i \leq j \leq r, \quad \text{and } \text{wt } \mathcal{P}^k = \text{wt } \mathcal{P}.$$

2.11. Partition overlaid patterns (POPs). A *partition overlaid pattern (POP)* consists of a GT pattern \mathcal{P} , and for every pair (i, j) of integers with $1 \leq i \leq j \leq r$, a partition $\underline{\pi(j)}^i$ that fits into the rectangle $(d_{i,j}(\mathcal{P}), d'_{i,j}(\mathcal{P}))$ (see [8] for more details). For $\lambda \in P^+$, let \mathbb{P}_λ denote the set of POPs with bounding sequence $\underline{\lambda}$.

The *bounding sequence*, *area* $\triangle(\mathfrak{P})$, *trapezoidal area* $\square(\mathfrak{P})$, *weight* $\text{wt } \mathfrak{P}$, and the *differences* $d_{i,j}(\mathfrak{P}), d'_{i,j}(\mathfrak{P})$ ($1 \leq i \leq j \leq r$) (or just $d_{i,j}$ and $d'_{i,j}$ if \mathfrak{P} is clear from the context) of a POP \mathfrak{P} are just the corresponding notions attached to the underlying pattern.

Fix $\lambda \in P^+$ and a POP \mathfrak{P} with bounding sequence $\underline{\lambda}$. Let $\underline{\lambda}^1, \dots, \underline{\lambda}^r, \underline{\lambda}^{r+1} = \underline{\lambda}$ be the underlying pattern of \mathfrak{P} and $\underline{\pi(j)}^i$, $1 \leq i \leq j \leq r$, be the partition overlay.

2.11.1. Restriction of a POP. For $1 \leq i \leq j \leq r+1$, define $\underline{\lambda}_i^j := \lambda_i^j, \lambda_{i+1}^j, \dots, \lambda_j^j$. Observe that $\underline{\lambda}_1^j = \underline{\lambda}^j$. For $s \in I \cup \{r+1\}$, the *restriction* \mathfrak{P}_s or $\text{res}_s(\mathfrak{P})$ of \mathfrak{P} is a POP with bounding sequence $\underline{\lambda}_s^{r+1}$: the rows of \mathfrak{P}_s are $\underline{\lambda}_s^s, \underline{\lambda}_s^{s+1}, \dots, \underline{\lambda}_s^{r+1}$ and $\underline{\pi(j)}^i$, $s \leq i \leq j \leq r$, be the partition overlay. Observe that $\mathfrak{P}_1 = \mathfrak{P}$.

2.11.2. Depth of a POP. For $1 \leq i \leq j \leq r$, set

$$d_i^j(\mathfrak{P}) := d_{i,j} \left(\sum_{p=i+1}^j d'_{p,j} \right) + |\underline{\pi(j)}^i|.$$

The *depth* $d(\mathfrak{P})$ of \mathfrak{P} is defined by:

$$d(\mathfrak{P}) := \sum_{1 \leq i \leq j \leq r} d_i^j(\mathfrak{P}).$$

We observe for $s \in I$ that

$$d(\mathfrak{P}_s) = \sum_{s \leq i \leq j \leq r} d_i^j(\mathfrak{P}) = d(\mathfrak{P}_{s+1}) + \sum_{j=s}^r d_s^j(\mathfrak{P}) = d(\mathfrak{P}_{s+1}) + \sum_{j=s+1}^r d_s^j(\mathfrak{P}) + |\underline{\pi(s)}^s|. \quad (2.9)$$

From [8, Corollary 3.4], we have the following:

$$\square(\mathfrak{P}) = \triangle(\mathfrak{P}) + d(\mathfrak{P}) - \sum_{1 \leq i \leq j \leq r} |\underline{\pi(j)}^i| = \frac{1}{2}((\lambda|\lambda) - (\text{wt } \mathfrak{P}|\text{wt } \mathfrak{P})). \quad (2.10)$$

2.11.3. Shift of a POP. For $k \in \mathbb{Z}_{\geq 0}$, the *shift* \mathfrak{P}^k of \mathfrak{P} by k is a POP with bounding sequence $\underline{\lambda} + k\underline{\theta}$: the underlying pattern of \mathfrak{P}^k is \mathcal{P}^k and $\underline{\pi(j)}^i$, $1 \leq i \leq j \leq r$, be the partition overlay. Note that the underlying partition overlays for \mathfrak{P}^k and \mathfrak{P} are same. It is easy to observe that

$$\text{wt } \mathfrak{P}^k = \text{wt } \mathfrak{P} \quad \text{and} \quad d(\mathfrak{P}^k) = d(\mathfrak{P}). \quad (2.11)$$

2.11.4. Shift and then restrict. For $k \in \mathbb{Z}_{\geq 0}$ and $s \in I \cup \{r+1\}$, set $\mathfrak{P}_s^k := \text{res}_s(\mathfrak{P}^k)$.

2.11.5. *Invariant set of a POP.* The invariant set $\mathcal{I}(\mathfrak{P})$ of \mathfrak{P} is defined by:

$$\mathcal{I}(\mathfrak{P}) := \{d_{i,j}(\mathfrak{P}), d'_{i,j}(\mathfrak{P}) : 1 \leq i < j \leq r\} \cup \{\underline{\pi(j)} : 1 \leq i \leq j \leq r\}.$$

Note for $s \in I \cup \{r+1\}$ that

$$\mathcal{I}(\mathfrak{P}_s) = \{d_{i,j}(\mathfrak{P}), d'_{i,j}(\mathfrak{P}) : s \leq i < j \leq r\} \cup \{\underline{\pi(j)} : s \leq i \leq j \leq r\}.$$

For $1 \leq s \leq j \leq r$, define

$$\mathcal{I}_s^j(\mathfrak{P}) := \{d_{s,j}(\mathfrak{P}), \underline{\pi(j)}^s\} \cup \{d'_{i,j}(\mathfrak{P}) : s < i \leq j\}.$$

Now we have

$$\mathcal{I}(\mathfrak{P}_s) = \mathcal{I}(\mathfrak{P}_{s+1}) \cup \left(\bigcup_{s \leq j \leq r} \mathcal{I}_s^j(\mathfrak{P}) \right), \quad \forall s \in I. \quad (2.12)$$

We observe that the invariant set of a POP is invariant under the shift, i.e.,

$$\mathcal{I}(\mathfrak{P}^k) = \mathcal{I}(\mathfrak{P}), \quad \text{for all } k \in \mathbb{Z}_{\geq 0}.$$

3. THE MAIN RESULT

3.1. **Bases for local Weyl modules in type A.** In this subsection, we recall the bases given by Chari and Loktev [2] in terms of POPs (see [8]). Fix notation and terminology as in §2.

3.1.1. Let d, d' be non-negative integers and $\underline{\pi}$ a partition that fits into the rectangle (d, d') . For an integer k with $1 \leq k \leq d'$, let $\mathbf{m}(d, d', \underline{\pi}, k)$ denote the number of parts of $\underline{\pi}$ that equal k . Set $\mathbf{m}(d, d', \underline{\pi}, 0) = d - \sum_{k=1}^{d'} \mathbf{m}(d, d', \underline{\pi}, k)$. For $\alpha \in R$, let $x_\alpha(d, d', \underline{\pi})$ denote

$$(x_\alpha \otimes 1)^{(\mathbf{m}(d, d', \underline{\pi}, d'))} (x_\alpha \otimes t^1)^{(\mathbf{m}(d, d', \underline{\pi}, d'-1))} \dots (x_\alpha \otimes t^{d'})^{(\mathbf{m}(d, d', \underline{\pi}, 0))}, \quad (3.1)$$

where, for an element X of $\mathfrak{g}[t]$ and a non-negative integer n , the symbol $X^{(n)}$ denotes the *divided power* $X^n/n!$. The order of factors in (3.1) is immaterial since they commute with each other, so we may simply write $x_\alpha(d, d', \underline{\pi}) = \prod_{k=0}^{d'} (x_\alpha \otimes t^k)^{(\mathbf{m}(d, d', \underline{\pi}, d'-k))}$.

3.1.2. Let $\lambda \in P^+$ and \mathfrak{P} be a POP with bounding sequence $\underline{\lambda}$. Let $\underline{\lambda}^1, \dots, \underline{\lambda}^r, \underline{\lambda}^{r+1} = \underline{\lambda}$ be the underlying pattern of \mathfrak{P} and $\underline{\pi(j)}^i$, $1 \leq i \leq j \leq r$, be the partition overlay. Now define $\rho_{\mathfrak{P}} \in \mathbf{U}(\mathfrak{n}^- \otimes \mathbb{C}[t])$ as follows:

$$\rho_{\mathfrak{P}} := x_{1,1}^-(d_{1,1}, d'_{1,1}, \underline{\pi(1)}^1) \left(\prod_{i=1}^2 x_{i,2}^-(d_{i,2}, d'_{i,2}, \underline{\pi(2)}^i) \right) \cdots \left(\prod_{i=1}^r x_{i,r}^-(d_{i,r}, d'_{i,r}, \underline{\pi(r)}^i) \right). \quad (3.2)$$

The order of the factors matters in the expressions for $\rho_{\mathfrak{P}}$. Since $[x_{i,j}^-, x_{p,q}^-] = 0, \forall 1 \leq i \leq p \leq q \leq j \leq r$, it is easy to see that

$$\rho_{\mathfrak{P}} = \left(\prod_{j=1}^r x_{1,j}^-(d_{1,j}, d'_{1,j}, \underline{\pi(j)}^1) \right) \cdots \left(\prod_{j=r-1}^r x_{r-1,j}^-(d_{r-1,j}, d'_{r-1,j}, \underline{\pi(j)}^{r-1}) \right) x_{r,r}^-(d_{r,r}, d'_{r,r}, \underline{\pi(r)}^r). \quad (3.3)$$

Set $\rho_{\mathfrak{P}_{r+1}} := 1$. We observe that

$$\rho_{\mathfrak{P}_s} = \left(\prod_{j=s}^r x_{s,j}^-(d_{s,j}, d'_{s,j}, \underline{\pi(j)}^s) \right) \rho_{\mathfrak{P}_{s+1}}, \quad \forall s \in I.$$

Define $v_{\mathfrak{P}} := \epsilon_{\mathfrak{P}} \rho_{\mathfrak{P}} w_{\lambda}$, where $\epsilon_{\mathfrak{P}} \in \{\pm 1\}$ is defined in §4.4.

The following theorem is proved in [2] (see [8, Theorem 4.5] for the current formulation).

Theorem 3.1. [2, 8] *The elements $v_{\mathfrak{P}}$, \mathfrak{P} belongs to the set \mathbb{P}_{λ} of POPs with bounding sequence $\underline{\lambda}$, form a basis for the local Weyl module $W(\lambda)$.*

We shall call the bases given in the last theorem as the *Chari-Loktev (or CL) bases*.

3.2. The main theorem: stability of the CL bases. We wish to study for $\lambda \in P^+$ and $k \in \mathbb{Z}_{\geq 0}$, the compatibility of CL bases with respect to the embeddings $W(\lambda) \hookrightarrow W(\lambda + k\theta)$ in $L(\Lambda_{i_{\lambda}})$ (see §2.7). We first recall the weight preserving embedding from \mathbb{P}_{λ} into $\mathbb{P}_{\lambda+k\theta}$ given in [8] at the level of the parametrizing sets of these bases: for $\mathfrak{P} \in \mathbb{P}_{\lambda}$, the shift \mathfrak{P}^k of \mathfrak{P} by k be its image in $\mathbb{P}_{\lambda+k\theta}$.

For every $\lambda \in P^+$, we will fix the following choice of w_{λ} in $L(\Lambda_{i_{\lambda}})$:

$$w_{\lambda} := T_{\lambda} v_{\Lambda_0},$$

where T_{λ} is a bijection from $L(\Lambda_0) \rightarrow L(\Lambda_{i_{\lambda}})$ defined in §4.3.

Lemma 3.2. *Let $\lambda \in P^+$ and $\mathfrak{P} \in \mathbb{P}_{\lambda}$. Then the weight of $v_{\mathfrak{P}}$ in $L(\Lambda_{i_{\lambda}})$ is*

$$t_{\text{wt } \mathfrak{P} - \bar{\Lambda}_{i_{\lambda}}}(\Lambda_{i_{\lambda}}) - d(\mathfrak{P})\delta,$$

where $\bar{\Lambda}_{i_{\lambda}}$ denotes the restriction to \mathfrak{h} of $\Lambda_{i_{\lambda}}$.

Proof. It is clear from the definition of $v_{\mathfrak{P}}$ that its weight in $L(\Lambda_{i_{\lambda}})$ is

$$\begin{aligned} & t_{\lambda}(\Lambda_0) - \sum_{1 \leq i \leq j \leq r} d_{i,j} \alpha_{i,j} + (\triangle(\mathfrak{P}) - \sum_{1 \leq i \leq j \leq r} |\underline{\pi(j)}^i|) \delta \\ &= \Lambda_0 + \text{wt } \mathfrak{P} - \left(\frac{1}{2}(\lambda|\lambda) - \triangle(\mathfrak{P}) + \sum_{1 \leq i \leq j \leq r} |\underline{\pi(j)}^i| \right) \delta \\ &= \Lambda_0 + \text{wt } \mathfrak{P} - \left(\frac{1}{2}(\text{wt } \mathfrak{P} | \text{wt } \mathfrak{P}) + d(\mathfrak{P}) \right) \delta \end{aligned} \quad (3.4)$$

where the last equality follows from (2.10). Since $\Lambda_{i_{\lambda}}$ is of level 1, we obtain using [6, (6.5.3)] that

$$t_{\text{wt } \mathfrak{P} - \bar{\Lambda}_{i_{\lambda}}}(\Lambda_{i_{\lambda}}) = \Lambda_0 + \text{wt } \mathfrak{P} + \frac{1}{2}((\Lambda_{i_{\lambda}} | \Lambda_{i_{\lambda}}) - (\text{wt } \mathfrak{P} | \text{wt } \mathfrak{P}))\delta. \quad (3.5)$$

We have $(\Lambda_{i_{\lambda}} | \Lambda_{i_{\lambda}}) = (w\Lambda_{i_{\lambda}} | w\Lambda_{i_{\lambda}}) = (t_{w_0\lambda}\Lambda_0 | t_{w_0\lambda}\Lambda_0) = 0$. Hence the result. \square

The following is immediate from Lemma 3.2 and (2.11).

Lemma 3.3. *Let $\lambda \in P^+$, $\mathfrak{P} \in \mathbb{P}_{\lambda}$, and $k \in \mathbb{Z}_{\geq 0}$. Then the basis vectors $v_{\mathfrak{P}} \in W(\lambda)$ and $v_{\mathfrak{P}^k} \in W(\lambda + k\theta)$ lie in the same weight space of $L(\Lambda_{i_{\lambda}})$.*

It is not true that $v_{\mathfrak{P}}$ and $v_{\mathfrak{P}^k}$ are equal as elements of $L(\Lambda_{i_\lambda})$ (see [7, Example 1]). We will however see below that $v_{\mathfrak{P}} = v_{\mathfrak{P}^k}$ for all *stable* \mathfrak{P} . More precisely, let

$$\mathbb{P}^{\text{stab}}(\lambda) := \{\mathfrak{P} \in \mathbb{P}_\lambda : d_{\ell,\ell}(\mathfrak{P}) \geq d(\mathfrak{P}_\ell), \forall 1 \leq \ell \leq r\} \text{ (see §§2.10 – 2.11).}$$

The following is the main result of this paper.

Theorem 3.4. *Let $\lambda \in P^+$ and $\mathfrak{P} \in \mathbb{P}^{\text{stab}}(\lambda)$. Then*

$$v_{\mathfrak{P}^k} = v_{\mathfrak{P}}, \quad \text{for all } k \in \mathbb{Z}_{\geq 0},$$

i.e., they are equal as elements of $L(\Lambda_{i_\lambda})$.

This theorem is proved in §4.

Remark 3.5. *Theorem 3.4 is conjectured in [8, §§6–7] and it is proved in $r = 1$ case in [7, Theorem 6] under the additional assumption that*

$$d(\mathfrak{P}) \leq \begin{cases} \min\{d_{1,1}, d'_{1,1}\}, & \lambda_1^2 \text{ even,} \\ \min\{d_{1,1}, d'_{1,1} - 1\}, & \lambda_1^2 \text{ odd.} \end{cases}$$

3.3. Bases for level one representations of $\widehat{\mathfrak{g}}$. For $i \in \widehat{I}$, consider $L(\Lambda_i)$, and let $t_\gamma(\Lambda_i) - d\delta$ be a weight of this module. Set $\mu = \varpi_i + \gamma$, the restriction of $t_\gamma(\Lambda_i) - d\delta$ to \mathfrak{h}^* . Let $\lambda \in P^+$ such that μ is a weight of the corresponding irreducible representation $V(\lambda)$ of \mathfrak{g} . Choose the tuple $\underline{\mu}$ corresponding to μ so that $\sum_{i=1}^{r+1} \mu_i = \sum_{i=1}^{r+1} \lambda_i$. Note that $i_\lambda = i$.

From Lemma 3.2, for $k \in \mathbb{Z}_{\geq 0}$, we get the CL basis indexing set for $W(\lambda + k\theta)_{t_\gamma(\Lambda_i) - d\delta}$ is the set $\mathbb{P}_{\lambda,\mu}^k(d)$ of POPs with bounding sequence $\underline{\lambda} + k\underline{\theta}$ with weight μ and depth d . From [8, Theorem 5.10], for $k \geq d$, there exist a bijection from the set $\mathcal{P}_r(d)$ of all r -colored partitions of d to $\mathbb{P}_{\lambda,\mu}^k(d)$. Since this bijection is produced by the “shift by k ” operator, we have

$$d_{\ell,\ell}(\mathfrak{P}) \geq k, \quad \forall 1 \leq \ell \leq r, \quad \text{for every } \mathfrak{P} \in \mathbb{P}_{\lambda,\mu}^k(d). \quad (3.6)$$

Thus from Proposition 2.2, for $k \geq d$, we have

$$W(\lambda + k\theta)_{t_\gamma(\Lambda_i) - d\delta} = L(\Lambda_i)_{t_\gamma(\Lambda_i) - d\delta},$$

and the set $\mathcal{B}_{\gamma,d} := \{v_{\mathfrak{P}} : \mathfrak{P} \in \mathbb{P}_{\lambda,\mu}^k(d)\}$ is a basis for $L(\Lambda_i)_{t_\gamma(\Lambda_i) - d\delta}$. By Theorem 3.4, using (3.6), the set $\mathcal{B}_{\gamma,d}$ is independent of the choice of k for any $k \geq d$. Finally, to obtain a basis for $L(\Lambda_i)$, we take the disjoint union over the weights of $L(\Lambda_i)$:

$$\mathcal{B} := \bigsqcup_{\gamma,d} \mathcal{B}_{\gamma,d}.$$

We may view \mathcal{B} as a direct limit of the CL bases for the Demazure submodules of $L(\Lambda_i)$.

3.4. An another set of bases and its stability. We note that for $\lambda \in P^+$, the generator w_λ of $W(\lambda) = V_{t_{w_0\lambda}}(\Lambda_0)$ is not a lowest weight vector of the Demazure module $V_{t_{w_0\lambda}}(\Lambda_0)$; while the lowest weight in $V_{t_{w_0\lambda}}(\Lambda_0)$ is $t_{w_0\lambda}(\Lambda_0)$, the weight of w_λ is in fact $t_\lambda(\Lambda_0)$. From the CL basis, it is easy to construct a basis consisting of monomials in the raising operators of the current algebra acting on a lowest weight vector v_λ of the Demazure module. Given a POP \mathfrak{P} with bounding sequence $\underline{\lambda}$, define the following element of $W(\lambda)$:

$$\bar{v}_{\mathfrak{P}} := x_{r,r}^+(d_{1,1}, d'_{1,1}, \underline{\pi(1)}^1) \left(\prod_{i=1}^2 x_{r-1, r+1-i}^+(d_{i,2}, d'_{i,2}, \underline{\pi(2)}^i) \right) \cdots \left(\prod_{i=1}^r x_{1, r+1-i}^+(d_{i,r}, d'_{i,r}, \underline{\pi(r)}^i) \right) v_\lambda.$$

Proposition 3.6. *The set $\{\bar{v}_{\mathfrak{P}} : \mathfrak{P} \in \mathbb{P}_\lambda\}$ is a basis for the local Weyl module $W(\lambda)$.*

The proof appears in §4.8. These bases also exhibits similar stabilization behavior as the CL bases.

4. PROOF OF THE MAIN RESULT

4.1. Frenkel-Kac translation operators. We recall the necessary facts from [5]. Let (V, π) be an integrable representation of $\widehat{\mathfrak{g}}$ with weight space decomposition $V = \bigoplus_{\mu \in \widehat{\mathfrak{h}}^*} V_\mu$. For a real root $\gamma = \alpha + s\delta$ ($\alpha \in R, s \in \mathbb{Z}$) of $\widehat{\mathfrak{g}}$ we define

$$r_\gamma^\pi := e^{-\pi(x_\gamma)} e^{\pi(x_{-\gamma})} e^{-\pi(x_\gamma)} \quad (\text{see (2.1)}). \quad (4.1)$$

The operator r_γ^π is a linear automorphism of V such that $r_\gamma^\pi(V_\mu) = V_{s_\gamma(\mu)}$, where $s_\gamma \in \widehat{W}$ is the reflection defined by γ . Given $w \in \widehat{W}$ and its reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_q}$, define

$$r_w^\pi := r_{\alpha_{i_1}}^\pi r_{\alpha_{i_2}}^\pi \cdots r_{\alpha_{i_q}}^\pi.$$

Note that $r_w^\pi(V_\mu) = V_{w\mu}$.

Next, we introduce the translation operators T_β on V for each $\beta \in Q$. For $\alpha \in R$, define

$$T_\alpha^\pi := r_{\delta-\alpha} r_\alpha. \quad (4.2)$$

For $\beta, \beta' \in Q$, set

$$T_\beta^\pi T_{\beta'}^\pi := \epsilon(\beta, \beta') T_{\beta+\beta'}^\pi,$$

where ϵ be a 2-cocycle of Q with values in $\{\pm 1\}$ (see [5, §2.3]). These operators satisfy $T_\beta(V_\mu) = V_{t_\beta(\mu)}$ for all $\mu \in \widehat{\mathfrak{h}}^*$, $\beta \in Q$.

We will only need these operators in two cases, namely when (V, π) is either the adjoint representation or the basic representation of $\widehat{\mathfrak{g}}$. We note that T_β^{ad} is in fact a Lie algebra automorphism of $\widehat{\mathfrak{g}}$. For ease of notation, we will denote the translation operators corresponding to the basic representation simply by T_β , suppressing the π in the superscript.

The key properties of the translation operators are given in [5, Propositions 1.2 and 2.3]. We summarize them for our context below:

Proposition 4.1. [5] *Let $\mu \in Q$. Then*

- (1) $T_\mu(h \otimes t^s)v = (h \otimes t^s)T_\mu v, \quad \forall h \in \mathfrak{h}, v \in L(\Lambda_0), s \in \mathbb{Z}.$
- (2) $T_{\mu-d\alpha} T_{d\alpha} = \epsilon(\mu - d\alpha, d\alpha)T_\mu, \quad \forall d \in \mathbb{Z}_{\geq 0}.$

$$(3) \quad T_\mu T_{-\mu} = \text{id}_{L(\Lambda_0)}.$$

$$(4) \quad T_{-\mu}(x_\alpha^- \otimes t^s) T_\mu v = (x_\alpha^- \otimes t^{s-(\mu|\alpha)}) v, \quad \forall \alpha \in R^+, v \in L(\Lambda_0), s \in \mathbb{Z}.$$

4.2. The goal of this subsection is to define an isomorphism T_{ϖ_i} associated to a fundamental weight ϖ_i ($i \in I$) of \mathfrak{g} .

4.2.1. Let τ be an automorphism of $\widehat{\mathfrak{g}}$ such that $\tau\widehat{\mathfrak{h}} = \widehat{\mathfrak{h}}$. We have the induced action of τ on $\widehat{\mathfrak{h}}^*$ by $\langle \tau\lambda, h \rangle = \langle \lambda, \tau^{-1}h \rangle$. Given an $\widehat{\mathfrak{g}}$ -module V , let V^τ denote the module with the twisted action

$$x \circ v = \tau^{-1}(x) v \text{ for } x \in \widehat{\mathfrak{g}}, v \in V.$$

Observe that for automorphisms τ_1, τ_2 , we have $V^{\tau_1\tau_2} \simeq (V^{\tau_2})^{\tau_1}$.

For $i \in I$, we now study the twisted actions on $L(\Lambda_0)$ by two specific automorphisms $\tilde{\sigma}_i, r_{w_0w_{0,i}}^{\text{ad}}$ of $\widehat{\mathfrak{g}}$. First, recall from §2 that $\sigma_i = t_{\varpi_i} w_{0,i} w_0 \in \widetilde{W}$ is an automorphism of the Dynkin diagram of $\widehat{\mathfrak{g}}$;

$$\sigma_i \alpha_p = \alpha_{i+p(\bmod r+1)}, \quad \forall p \in \widehat{I}, \quad \text{and } \sigma \rho = \rho.$$

Consider the Lie algebra automorphism $\tilde{\sigma}_i$ of $\widehat{\mathfrak{g}}$ given by the relations

$$\tilde{\sigma}_i(e_p) = e_{i+p(\bmod r+1)}, \quad \tilde{\sigma}_i(f_p) = f_{i+p(\bmod r+1)}, \quad \tilde{\sigma}_i(\alpha_p^\vee) = \alpha_{i+p(\bmod r+1)}^\vee, \quad \forall p \in \widehat{I}, \quad \text{and } \tilde{\sigma}_i(\rho^\vee) = \rho^\vee. \quad (4.3)$$

Here $\rho^\vee \in \widehat{\mathfrak{h}}$ is the unique element for which $\langle \alpha_p, \rho^\vee \rangle = 1, \forall p \in \widehat{I}$, and $\langle \Lambda_0, \rho^\vee \rangle = 0$. Clearly $\tilde{\sigma}_i$ (resp. $r_{w_0w_{0,i}}^{\text{ad}}$) leaves $\widehat{\mathfrak{h}}$ invariant, and its induced action on $\widehat{\mathfrak{h}}^*$ coincides with σ_i (resp. $w_0w_{0,i}$), i.e.,

$$\langle \sigma_i \Lambda, h \rangle = \langle \Lambda, \tilde{\sigma}_i^{-1} h \rangle \quad \text{and} \quad \langle w_0w_{0,i} \Lambda, h \rangle = \langle \Lambda, (r_{w_0w_{0,i}}^{\text{ad}})^{-1} h \rangle, \quad \forall h \in \widehat{\mathfrak{h}}, \Lambda \in \widehat{\mathfrak{h}}^*. \quad (4.4)$$

It is well-known that

$$r_{w_0w_{0,i}}^{\text{ad}}(x_\alpha \otimes t^s) = x_{w_0w_{0,i}(\alpha)} \otimes t^s \quad \text{and} \quad (r_{w_0w_{0,i}}^{\text{ad}})^{-1}(x_\alpha \otimes t^s) = x_{w_{0,i}w_0(\alpha)} \otimes t^s, \quad \forall \alpha \in R, s \in \mathbb{Z}. \quad (4.5)$$

Note for $p \in \widehat{I}$ that

$$w_0w_{0,i}(\alpha_p) = \begin{cases} \alpha_{r+1+p-i}, & p < i, \\ \alpha_{p-i}, & p > i, \\ -\theta, & p = i, \\ \alpha_{r+1-i} + \delta, & p = 0. \end{cases} \quad \text{and} \quad w_{0,i}w_0(\alpha_p) = \begin{cases} \alpha_{p+i-r-1}, & r+1-p < i, \\ \alpha_{p+i}, & r+1-p > i, \\ -\theta, & r+1-p = i, \\ \alpha_i + \delta, & p = 0. \end{cases} \quad (4.6)$$

Set $\tilde{t}_{\varpi_i} := \tilde{\sigma}_i r_{w_0w_{0,i}}^{\text{ad}}$. Observe that $\tilde{t}_{\varpi_i}(\mathfrak{g}_\alpha) = \mathfrak{g}_{t_{\varpi_i}(\alpha)}, \forall \alpha \in R$. From (4.4), we have

$$\langle t_{\varpi_i} \Lambda, h \rangle = \langle \Lambda, \tilde{t}_{\varpi_i}^{-1} h \rangle, \quad \forall h \in \widehat{\mathfrak{h}}, \Lambda \in \widehat{\mathfrak{h}}^*. \quad (4.7)$$

Using (4.3), (4.5), and (4.6), we get for $p \in \widehat{I}$ that

$$\tilde{t}_{\varpi_i}(e_p) = \begin{cases} e_p, & p \neq 0, i, \\ e_i \otimes t^{-1}, & p = i, \\ x_{-\theta} \otimes t^2, & p = 0, \end{cases} \quad \tilde{t}_{\varpi_i}(f_p) = \begin{cases} f_p, & p \neq 0, i, \\ f_i \otimes t, & p = i, \\ x_\theta \otimes t^{-2}, & p = 0, \end{cases} \quad (4.8)$$

and

$$\tilde{t}_{\varpi_i}^{-1}(e_p) = \begin{cases} e_p, & p \neq 0, i, \\ e_i \otimes t, & p = i, \\ x_{-\theta}, & p = 0, \end{cases} \quad \tilde{t}_{\varpi_i}^{-1}(f_p) = \begin{cases} f_p, & p \neq 0, i, \\ f_i \otimes t^{-1}, & p = i, \\ x_\theta, & p = 0. \end{cases} \quad (4.9)$$

Thus

$$\tilde{t}_{\varpi_i}(x_\alpha \otimes t^s) = (x_\alpha \otimes t^{s-(\varpi_i|\alpha)}) \quad \text{and} \quad \tilde{t}_{\varpi_i}^{-1}(x_\alpha \otimes t^s) = (x_\alpha \otimes t^{s+(\varpi_i|\alpha)}), \quad \forall \alpha \in R, s \in \mathbb{Z}. \quad (4.10)$$

Proposition 4.2. *With notation as above, for $i \in I$, we have $L(\Lambda_0)^{\tilde{t}_{\varpi_i}} \simeq L(\Lambda_i)$.*

Proof. We consider the $\mathbf{U}(\widehat{\mathfrak{g}})$ -linear map $L(\Lambda_i) \rightarrow L(\Lambda_0)^{\tilde{t}_{\varpi_i}}$ which sends v_{Λ_i} to v_{Λ_0} . To show this is well defined, we only need to check that $v_{\Lambda_0} \in L(\Lambda_0)^{\tilde{t}_{\varpi_i}}$ satisfies the relations (2.3)-(2.5) for $\Lambda = \Lambda_i$. Since $w_0 w_{0,i} \Lambda_0 = \Lambda_0$, the relation (2.3) follows from (4.4). The relation (2.4) is immediate from (4.9). To prove the relation (2.5), using (4.9), we only need to show that $(f_i \otimes t^{-1})^2 v_{\Lambda_0} = 0$ in $L(\Lambda_0)$. But this follows easily by a standard \mathfrak{sl}_2 argument using the \mathfrak{sl}_2 copy spanned by $e_i \otimes t, f_i \otimes t^{-1}$, and $\alpha_i^\vee + c$. Now, this map is a surjection, since v_{Λ_0} generates $L(\Lambda_0)^{\tilde{t}_{\varpi_i}}$. Since $L(\Lambda_i)$ is irreducible, it must be an isomorphism. \square

For $i \in I$, let T_{ϖ_i} be the isomorphism from $L(\Lambda_0)^{\tilde{t}_{\varpi_i}}$ onto $L(\Lambda_i)$. Note that

$$T_{\varpi_i} L(\Lambda_0)_\nu = L(\Lambda_{i_\lambda})_{t_{\varpi_i}(\nu)}, \quad \forall \nu \in \widehat{\mathfrak{h}}^*.$$

Set $T_{-\varpi_i} := T_{\varpi_i}^{-1}$. The isomorphism $T_{-\varpi_i} : L(\Lambda_i) \rightarrow L(\Lambda_0)^{\tilde{t}_{\varpi_i}}$ maps $v_{\Lambda_i} \mapsto v_{\Lambda_0}$. It is then determined on all of $L(\Lambda_i)$ by $\widehat{\mathfrak{g}}$ -linearity, i.e., by the relation

$$T_{-\varpi_i}(xv) = \tilde{t}_{\varpi_i}^{-1}(x) T_{-\varpi_i}(v), \quad \forall x \in \widehat{\mathfrak{g}}, v \in L(\Lambda_i). \quad (4.11)$$

Now using (4.10), we get

$$T_{-\varpi_i}(x_\alpha^- \otimes t^s) T_{\varpi_i} v = (x_\alpha^- \otimes t^{s-(\varpi_i|\alpha)}) v, \quad \forall \alpha \in R^+, v \in L(\Lambda_0), s \in \mathbb{Z}. \quad (4.12)$$

4.3. For $\lambda \in P^+$ and $\beta \in Q$, we define a bijection $T_{\lambda-\beta} : L(\Lambda_0) \rightarrow L(\Lambda_{i_\lambda})$ as follows:

$$T_{\lambda-\beta} := T_{\varpi_{i_\lambda}} T_{\lambda-\beta-\varpi_{i_\lambda}}.$$

Suppose there is $\lambda' \in P^+$ and $\beta' \in Q$ such that $\lambda - \beta = \lambda' - \beta'$. Then it is easy to see that $i_\lambda = i_{\lambda'}$. Hence the definition is well defined. We observe that

$$T_{\lambda-\beta} L(\Lambda_0)_\nu = L(\Lambda_{i_\lambda})_{t_{\lambda-\beta}(\nu)}, \quad \forall \nu \in \widehat{\mathfrak{h}}^*.$$

Set $T_{-(\lambda-\beta)} := T_{\lambda-\beta}^{-1}$.

Proposition 4.3. *Let $\lambda \in P^+$ and $\beta \in Q$. Then*

- (1) $T_{\lambda-\beta-d\alpha} T_{d\alpha} = T_{\lambda-\beta}, \quad \forall d \in \mathbb{Z}_{\geq 0}.$
- (2) $T_{-(\lambda-\beta)}(x_\alpha^- \otimes t^s) T_{\lambda-\beta} v = (x_\alpha^- \otimes t^{s-(\lambda-\beta|\alpha)}) v, \quad \forall \alpha \in R^+, v \in L(\Lambda_0), s \in \mathbb{Z}.$

Proof. The proof is immediate from parts (2) and (4) of Propositions 4.1, and (4.12). \square

4.4. Given $\lambda \in P^+$, $\mathfrak{P} \in \mathbb{P}_\lambda$, and $s \in I \cup \{r+1\}$, define $\epsilon_{\mathfrak{P}_s} \in \{\pm 1\}$ recursively as follows:

$$\epsilon_{\mathfrak{P}_{r+1}} := 1 \quad \text{and} \quad \epsilon_{\mathfrak{P}_s} := (-1)^{\lfloor \frac{d_{s,s}(\mathfrak{P})}{2} \rfloor} \epsilon \left(\lambda - \sum_{s \leq i \leq j \leq r} d_{i,j}(\mathfrak{P}) \alpha_{i,j}, d_{s,s}(\mathfrak{P}) \alpha_{s,s} \right) \epsilon_{\mathfrak{P}_{s+1}}, \quad \forall s \in I.$$

Here, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x .

We are now in a position to state the main result of this section.

Theorem 4.4. *Let $\lambda \in P^+$, $\mathfrak{P} \in \mathbb{P}_\lambda$, $k \in \mathbb{Z}_{\geq 0}$, and $s \in I \cup \{r+1\}$. If $d_{\ell,\ell}(\mathfrak{P}) \geq d(\mathfrak{P}_\ell)$ for all $s \leq \ell \leq r$, then*

$$\epsilon_{\mathfrak{P}_s^k} \rho_{\mathfrak{P}_s^k} T_{\lambda+k\theta} v_{\Lambda_0} = T_{\lambda+k\alpha_{1,s-1}-\sum_{s \leq i \leq j \leq r} d_{i,j}(\mathfrak{P}) \alpha_{i,j}} f_{\mathcal{I}(\mathfrak{P}_s)} v_{\Lambda_0}, \quad (4.13)$$

where $f_{\mathcal{I}(\mathfrak{P}_s)}$ is a polynomial in $\alpha_i^\vee t^{-j}$, $i \in I, j \in \mathbb{N}$, depends only on the elements of the set $\mathcal{I}(\mathfrak{P}_s)$, such that the weight of $f_{\mathcal{I}(\mathfrak{P}_s)} v_{\Lambda_0}$ in $L(\Lambda_0)$ is $\Lambda_0 - d(\mathfrak{P}_s)\delta$.

We observe that the expression on the left hand side of (4.13) depends on k , the one on the right hand side, when $s = 1$, is independent of it. The fact that these two expressions are equal when $d_{\ell,\ell}(\mathfrak{P}) \geq d(\mathfrak{P}_\ell)$ for all $1 \leq \ell \leq r$, what leads to the stability properties of interest. Thus Theorem 4.4 for $s = 1$, proves Theorem 3.4.

The proof of Theorem 4.4 occupies the rest of this section.

4.5. In this subsection, for $x \in \mathfrak{g}$, $s \in \mathbb{Z}$, and $m \in \mathbb{N}$, we set $xt^s := x \otimes t^s$ and $[m] := \{1, 2, \dots, m\}$.

Lemma 4.5. *Let $\alpha, \beta_1, \dots, \beta_n \in R^+$, $p, q_1, \dots, q_n \in \mathbb{Z}_{\geq 0}$, and $y_\alpha \in \mathfrak{g}_{-\alpha}$. Then*

$$(y_\alpha t^p) \left(\prod_{i=1}^n \beta_i^\vee t^{-q_i} \right) = \sum_{0 \leq k \leq n} \sum_{\substack{A \subseteq [n] \\ |A|=k}} \left(\prod_{i \in A} \langle \alpha, \beta_i^\vee \rangle \right) \left(\prod_{i \in [n] \setminus A} \beta_i^\vee t^{-q_i} \right) (y_\alpha t^{p - \sum_{i \in A} q_i}).$$

Proof. Proceed by induction on n . In case $n = 1$, we have

$$(y_\alpha t^p) (\beta_1^\vee t^{-q_1}) = (\beta_1^\vee t^{-q_1}) (y_\alpha t^p) + [y_\alpha t^p, \beta_1^\vee t^{-q_1}] = (\beta_1^\vee t^{-q_1}) (y_\alpha t^p) + \langle \alpha, \beta_1^\vee \rangle (y_\alpha t^{p-q_1}),$$

and the result is obvious. Now suppose that $n \geq 2$. Since $[y_\alpha t^p, \beta_n^\vee t^{-q_n}] = \langle \alpha, \beta_n^\vee \rangle (y_\alpha t^{p-q_n})$, we have

$$(y_\alpha t^p) \left(\prod_{i=1}^n \beta_i^\vee t^{-q_i} \right) = ((\beta_n^\vee t^{-q_n}) (y_\alpha t^p) + \langle \alpha, \beta_n^\vee \rangle (y_\alpha t^{p-q_n})) \left(\prod_{i=1}^{n-1} \beta_i^\vee t^{-q_i} \right).$$

Using induction hypothesis the right hand side of the last equation becomes

$$\begin{aligned} & (\beta_n^\vee t^{-q_n}) \sum_{0 \leq k' \leq n-1} \sum_{\substack{A' \subseteq [n-1] \\ |A'|=k'}} \left(\prod_{i \in A'} \langle \alpha, \beta_i^\vee \rangle \right) \left(\prod_{i \in [n-1] \setminus A'} \beta_i^\vee t^{-q_i} \right) (y_\alpha t^{p - \sum_{i \in A'} q_i}) \\ & + \langle \alpha, \beta_n^\vee \rangle \sum_{0 \leq k'' \leq n-1} \sum_{\substack{A'' \subseteq [n-1] \\ |A''|=k''}} \left(\prod_{i \in A''} \langle \alpha, \beta_i^\vee \rangle \right) \left(\prod_{i \in [n-1] \setminus A''} \beta_i^\vee t^{-q_i} \right) (y_\alpha t^{p-q_n - \sum_{i \in A''} q_i}). \end{aligned}$$

This completes the proof. Indeed, for any $A \subseteq [n]$, there exists $B \subseteq [n-1]$ such that either $A = B$ or $A = B \cup \{n\}$. \square

Lemma 4.6. *Let $\alpha, \beta_1, \dots, \beta_n \in R^+$, $p_1, \dots, p_m, q_1, \dots, q_n \in \mathbb{Z}_{\geq 0}$, and $y_\alpha \in \mathfrak{g}_{-\alpha}$. Then*

$$\begin{aligned} & \left(\prod_{i=1}^m y_\alpha t^{p_i} \right) \left(\prod_{i=1}^n \beta_i^\vee t^{-q_i} \right) \\ &= \sum_{\substack{k_1, \dots, k_m \\ 0 \leq k_i \leq n - \sum_{j=i+1}^m k_j}} \sum_{\substack{A_1, \dots, A_m \\ A_i \subseteq [n] \setminus \bigcup_{j=i+1}^m A_j \\ |A_i| = k_i}} \left(\prod_{i \in \bigcup_{j=1}^m A_j} \langle \alpha, \beta_i^\vee \rangle \right) \left(\prod_{i \in [n] \setminus \bigcup_{j=1}^m A_j} \beta_i^\vee t^{-q_i} \right) \left(\prod_{j=1}^m y_\alpha t^{p_j - \sum_{i \in A_j} q_i} \right). \end{aligned}$$

Proof. Proceed by induction on m . In case $m = 1$, we have the result from Lemma 4.5. Now suppose that $m \geq 2$. Using Lemma 4.5, we have

$$\begin{aligned} & \left(\prod_{i=1}^m y_\alpha t^{p_i} \right) \left(\prod_{i=1}^n \beta_i^\vee t^{-q_i} \right) \\ &= \left(\prod_{i=1}^{m-1} y_\alpha t^{p_i} \right) \sum_{0 \leq k_m \leq n} \sum_{\substack{A_m \subseteq [n] \\ |A_m| = k_m}} \left(\prod_{i \in A_m} \langle \alpha, \beta_i^\vee \rangle \right) \left(\prod_{i \in [n] \setminus A_m} \beta_i^\vee t^{-q_i} \right) (y_\alpha t^{p_m - \sum_{i \in A_m} q_i}). \end{aligned} \quad (4.14)$$

Using induction hypothesis, we have

$$\begin{aligned} & \left(\prod_{i=1}^{m-1} y_\alpha t^{p_i} \right) \left(\prod_{i \in [n] \setminus A_m} \beta_i^\vee t^{-q_i} \right) \\ &= \sum_{\substack{k_1, \dots, k_{m-1} \\ 0 \leq k_i \leq n - \sum_{j=i+1}^{m-1} k_j}} \sum_{\substack{A_1, \dots, A_{m-1} \\ A_i \subseteq [n] \setminus \bigcup_{j=i+1}^{m-1} A_j \\ |A_i| = k_i}} \left(\prod_{i \in \bigcup_{j=1}^{m-1} A_j} \langle \alpha, \beta_i^\vee \rangle \right) \left(\prod_{i \in [n] \setminus \bigcup_{j=1}^{m-1} A_j} \beta_i^\vee t^{-q_i} \right) \left(\prod_{j=1}^{m-1} y_\alpha t^{p_j - \sum_{i \in A_j} q_i} \right). \end{aligned} \quad (4.15)$$

Substituting (4.15) in the right hand side of (4.14), we get the result. \square

4.6. The following result follows from [7, Theorem 10 and Proposition 13 (1)].

Theorem 4.7. [7] *Let $\alpha \in R^+$. Let $d \in \mathbb{Z}_{\geq 0}$ and $\underline{\pi}$ be a partition such that $d \geq |\underline{\pi}|$. Then*

$$x_{-\alpha}(d, d, \underline{\pi}) T_{d\alpha} v_{\Lambda_0} = (-1)^{[\frac{d}{2}]} f_{\underline{\pi}} v_{\Lambda_0},$$

where $f_{\underline{\pi}}$ is a polynomial in $\alpha^\vee t^{-j}$, $j \in \mathbb{N}$, depends only on $\underline{\pi}$ and not on d , such that $f_{\underline{\pi}} v_{\Lambda_0}$ belongs to the weight space of $L(\Lambda_0)$ of weight $\Lambda_0 - |\underline{\pi}| \delta$.

Proposition 4.8. *Let $d, d', m \in \mathbb{Z}_{\geq 0}$, $\alpha \in R^+$, and $\underline{\pi}$ be a partition. Let $\lambda \in P^+$, and $\beta \in Q$ with $(\lambda - \beta|\alpha) = d + d'$, and set $\mu = \lambda - \beta$. Let g_m be a polynomial in $\alpha_i^\vee t^{-j}$, $i \in I, j \in \mathbb{N}$, such that the weight of $g_m v_{\Lambda_0}$ in $L(\Lambda_0)$ is $\Lambda_0 - m\delta$. Then*

(1) *the weight of $x_{-\alpha}(d, d', \underline{\pi}) T_\mu g_m v_{\Lambda_0}$ in $L(\Lambda_{i_\lambda})$ is $t_{\mu-d\alpha}(\Lambda_0 - (|\underline{\pi}| + m)\delta)$.*

(2) If $d \geq |\underline{\pi}| + m$, we have

$$x_{-\alpha}(d, d', \underline{\pi}) T_{\mu} g_m v_{\Lambda_0} = (-1)^{\lfloor \frac{d}{2} \rfloor} \epsilon(\mu - d\alpha, d\alpha) T_{\mu-d\alpha} f_{\underline{\pi}, g_m} v_{\Lambda_0},$$

where $f_{\underline{\pi}, g_m}$ is a polynomial in $\alpha_i^{\vee} t^{-j}$, $i \in I, j \in \mathbb{N}$, depends only on $\underline{\pi}$, g_m and not on d, d' , such that the weight of $f_{\underline{\pi}, g_m} v_{\Lambda_0}$ in $L(\Lambda_{i_{\lambda}})$ is $\Lambda_0 - (|\underline{\pi}| + m)\delta$.

Proof. Since the weight of $T_{\mu} g_m v_{\Lambda_0}$ is $t_{\mu}(\Lambda_0 - m\delta)$, we have the weight of $x_{-\alpha}(d, d', \underline{\pi}) T_{\mu} g_m v_{\Lambda_0}$ is $t_{\mu}(\Lambda_0 - m\delta) - d\alpha + (dd' - |\underline{\pi}|)\delta = t_{\mu-d\alpha}(\Lambda_0 - (|\underline{\pi}| + m)\delta)$. Hence part (1). We now prove part (2). Using Proposition 4.1, we have

$$\begin{aligned} \left(\prod_{i=1}^d y_{\alpha} t^{d' - \pi_i} \right) T_{\mu} g_m v_{\Lambda_0} &= \epsilon(\mu - d\alpha, d\alpha) T_{\mu-d\alpha} \left(\prod_{i=1}^d T_{-(\mu-d\alpha)} y_{\alpha} t^{d' - \pi_i} T_{\mu-d\alpha} \right) T_{d\alpha} g_m v_{\Lambda_0} \\ &= \epsilon(\mu - d\alpha, d\alpha) T_{\mu-d\alpha} \left(\prod_{i=1}^d y_{\alpha} t^{d' - \pi_i} \right) g_m T_{d\alpha} v_{\Lambda_0} \end{aligned} \quad (4.16)$$

Using Lemma 4.6, the right hand side of the last equation becomes

$$\epsilon(\mu - d\alpha, d\alpha) T_{\mu-d\alpha} \sum_j f_{\underline{\pi}, g_m}^j \left(\prod_{i=1}^d y_{\alpha} t^{d' - \eta_{i,j}} \right) T_{d\alpha} v_{\Lambda_0},$$

for some polynomials $f_{\underline{\pi}, g_m}^j$ in $\alpha_i^{\vee} t^{-j}$, $i \in I, j \in \mathbb{N}$, and positive integers $\eta_{i,j}$, depend on $\underline{\pi}, g_m$, such that $|\underline{\pi}| + m \geq \sum_j \eta_{i,j}$, $\forall 1 \leq i \leq d$. The result now follows from Theorem 4.7 and part (1). \square

Proposition 4.9. Let $\lambda \in P^+$, $\mathfrak{P} \in \mathbb{P}_{\lambda}$, $k, m \in \mathbb{Z}_{\geq 0}$, and $s \in I$. Let g_m be a polynomial in $\alpha_i^{\vee} t^{-j}$, $i \in I, j \in \mathbb{N}$, such that the weight of $g_m v_{\Lambda_0}$ in $L(\Lambda_0)$ is $\Lambda_0 - m\delta$. Then for every $s < q \leq r+1$, we have

$$\begin{aligned} &\left(\prod_{j=q}^r x_{s,j}^{-}(d_{s,j}, d'_{s,j} + \delta_{1,s} k, \underline{\pi}(j)^s) \right) T_{\lambda+k\alpha_{1,s} - \sum_{s < i \leq j \leq r} d_{i,j} \alpha_{i,j}} g_m v_{\Lambda_0} \\ &= T_{\lambda+k\alpha_{1,s} - \sum_{s < i \leq j \leq r} d_{i,j} \alpha_{i,j} - \sum_{j=q}^r d_{s,j} \alpha_{s,j}} f_{q, g_m} v_{\Lambda_0}, \end{aligned} \quad (4.17)$$

where f_{q, g_m} is a polynomial in $\alpha_i^{\vee} t^{-j}$, $i \in I, j \in \mathbb{N}$, depends only on g_m and the elements of the sets $\mathcal{I}_s^j(\mathfrak{P})$, $q \leq j \leq r$, such that the weight of $f_{q, g_m} v_{\Lambda_0}$ in $L(\Lambda_0)$ is $\Lambda_0 - (m + \sum_{j=q}^r d_s^j(\mathfrak{P}))\delta$.

Proof. Proceed by induction on q . In the case $q = r+1$, by taking $f_{r+1, g_m} = g_m$, both sides are equal to $T_{\lambda+k\alpha_{1,s} - \sum_{s < i \leq j \leq r} d_{i,j} \alpha_{i,j}} g_m v_{\Lambda_0}$. Now suppose that $q \leq r$. By the induction hypothesis, we have

$$\begin{aligned} &\left(\prod_{j=q+1}^r x_{s,j}^{-}(d_{s,j}, d'_{s,j} + \delta_{1,s} k, \underline{\pi}(j)^s) \right) T_{\lambda+k\alpha_{1,s} - \sum_{s < i \leq j \leq r} d_{i,j} \alpha_{i,j}} g_m v_{\Lambda_0} \\ &= T_{\lambda+k\alpha_{1,s} - \sum_{s < i \leq j \leq r} d_{i,j} \alpha_{i,j} - \sum_{j=q+1}^r d_{s,j} \alpha_{s,j}} f_{q+1, g_m} v_{\Lambda_0}, \end{aligned} \quad (4.18)$$

where f_{q+1, g_m} is a polynomial in $\alpha_i^{\vee} t^{-j}$, $i \in I, j \in \mathbb{N}$, depends only on g_m and the elements of the sets $\mathcal{I}_s^j(\mathfrak{P})$, $q+1 \leq j \leq r$, such that the weight of $f_{q+1, g_m} v_{\Lambda_0}$ in $L(\Lambda_0)$ is $\Lambda_0 - (m + \sum_{j=q+1}^r d_s^j(\mathfrak{P}))\delta$.

Acting both sides of (4.18) with $x_{s,q}^-(d_{s,q}, d'_{s,q} + \delta_{1,s}k, \underline{\pi(q)}^s)$, we get

$$\begin{aligned} & \left(\prod_{j=q}^r x_{s,j}^-(d_{s,j}, d'_{s,j} + \delta_{1,s}k, \underline{\pi(j)}^s) \right) T_{\lambda+k\alpha_{1,s}-\sum_{s< i \leq j \leq r} d_{i,j}\alpha_{i,j}} g_m v_{\Lambda_0} \\ &= x_{s,q}^-(d_{s,q}, d'_{s,q} + \delta_{1,s}k, \underline{\pi(q)}^s) T_{\lambda+k\alpha_{1,s}-\sum_{s< i \leq j \leq r} d_{i,j}\alpha_{i,j}-\sum_{j=q+1}^r d_{s,j}\alpha_{s,j}} f_{q+1,g_m} v_{\Lambda_0}. \end{aligned} \quad (4.19)$$

Set $\mu = \lambda + k\alpha_{1,s} - \sum_{s< i \leq j \leq r} d_{i,j}\alpha_{i,j} - \sum_{j=q}^r d_{s,j}\alpha_{s,j}$. We observe that

$$\begin{aligned} (\mu|\alpha_{s,q}) &= (\lambda_s^{r+1} - \lambda_{q+1}^{r+1}) + \delta_{1,s}k + \sum_{j=q+1}^r d_{q+1,j} - \sum_{i=s+1}^q d_{i,q} - \sum_{j=q}^r d_{s,j} - d_{s,q} \\ &= (\lambda_s^{r+1} - \lambda_{q+1}^{r+1}) + \delta_{1,s}k + (\lambda_{q+1}^{r+1} - \lambda_{q+1}^{q+1}) - (\lambda_s^q - \sum_{i=s}^q d'_{i,q} - \lambda_{q+1}^{q+1}) - (\lambda_s^{r+1} - \lambda_s^q) - d_{s,q} \\ &= \sum_{i=s}^q d'_{i,q} + \delta_{1,s}k - d_{s,q}. \end{aligned} \quad (4.20)$$

Using Proposition 4.1 and (4.20), the right hand side of (4.19) becomes

$$\begin{aligned} & z_{s,q} T_{\mu} \left(\prod_{p=0}^{d_{s,q}} T_{-\mu} \left(x_{s,q}^- \otimes t^{d'_{s,q} + \delta_{1,s}k - \pi(q)_p^s} \right) T_{\mu} \right) T_{d_{s,q}\alpha_{s,q}} f_{q+1,g_m} v_{\Lambda_0} \\ &= z_{s,q} T_{\mu} \left(\prod_{p=0}^{d_{s,q}} \left(x_{s,q}^- \otimes t^{d_{s,q} - \pi(q)_p^s - \sum_{i=s+1}^q d'_{i,q}} \right) \right) T_{d_{s,q}\alpha_{s,q}} f_{q+1,g_m} v_{\Lambda_0}. \end{aligned} \quad (4.21)$$

where $z_{s,q} := \prod_{\ell=0}^{d'_{s,q}} \frac{1}{m(d_{s,q}, d'_{s,q}, \underline{\pi(q)}^s, \ell)!}$. From Theorem 2.3 and Proposition 4.9 (1), we have

$$z_{s,q} \left(\prod_{p=0}^{d_{s,q}} \left(x_{s,q}^- \otimes t^{d_{s,q} - \pi(q)_p^s - \sum_{i=s+1}^q d'_{i,q}} \right) \right) T_{d_{s,q}\alpha_{s,q}} f_{q+1,g_m} v_{\Lambda_0} = f_{q,g_m} v_{\Lambda_0}, \quad (4.22)$$

where f_{q,g_m} is a polynomial in $\alpha_i^{\vee} t^{-j}$, $i \in I, j \in \mathbb{N}$, depends only on f_{q+1,g_m} and the elements of the set $\{d_{s,q}, d'_{i,q}, \underline{\pi(q)}^s : s < i \leq q\} = \mathcal{I}_s^q(\mathfrak{P})$, such that the weight of $f_{q,g_m} v_{\Lambda_0}$ in $L(\Lambda_0)$ is

$$\Lambda_0 - (|\underline{\pi(q)}^s| + d_{s,q} \left(\sum_{i=s+1}^q d'_{i,q} \right) + m + \sum_{j=q+1}^r d_s^j(\mathfrak{P})) \delta = \Lambda_0 - \left(m + \sum_{j=q}^r d_s^j(\mathfrak{P}) \right) \delta.$$

Substituting (4.22) into (4.21), we get the result. \square

4.7. Proof of Theorem 4.4. Proceed by induction on s . In the case $s = r + 1$, by taking $f_{\mathcal{I}(\mathfrak{P}_{r+1})} = 1$, both sides of (4.13) are equal to $T_{\lambda+k\theta} v_{\Lambda_0}$. Now suppose that $s \leq r$. By induction hypothesis, we have

$$\rho_{\mathfrak{P}_{s+1}}^k T_{\lambda+k\theta} v_{\Lambda_0} = \epsilon_{s+1} T_{\lambda+k\alpha_{1,s}-\sum_{s< i \leq j \leq r} d_{i,j}\alpha_{i,j}} f_{\mathcal{I}(\mathfrak{P}_{s+1})} v_{\Lambda_0}, \quad (4.23)$$

where $\epsilon_{s+1} = \epsilon_{\mathfrak{P}_{s+1}^k}$ and $f_{\mathcal{I}(\mathfrak{P}_{s+1})}$ is a polynomial in $\alpha_i^\vee t^{-j}$, $i \in I, j \in \mathbb{N}$, depends only on the elements of the set $\mathcal{I}(\mathfrak{P}_{s+1})$, such that the weight of $f_{\mathcal{I}(\mathfrak{P}_{s+1})} v_{\Lambda_0}$ in $L(\Lambda_0)$ is $\Lambda_0 - d(\mathfrak{P}_{s+1})\delta$. Since

$$\rho_{\mathfrak{P}_s^k} = \prod_{j=s}^r x_{s,j}^-(d_{s,j} + \delta_{s,j}k, d'_{s,j} + \delta_{1,s}k, \underline{\pi(j)}^s) \rho_{\mathfrak{P}_{s+1}^k},$$

we get from (4.23) that

$$\rho_{\mathfrak{P}_s^k} T_{\lambda+k\theta} v_{\Lambda_0} = \epsilon_{s+1} \left(\prod_{j=s}^r x_{s,j}^-(d_{s,j} + \delta_{s,j}k, d'_{s,j} + \delta_{1,s}k, \underline{\pi(j)}^s) \right) T_{\lambda+k\alpha_{1,s} - \sum_{s < i \leq j \leq r} d_{i,j}\alpha_{i,j}} f_{\mathcal{I}(\mathfrak{P}_{s+1})} v_{\Lambda_0}.$$

Now using Proposition 4.9 with $q = s + 1$, we get

$$\rho_{\mathfrak{P}_s^k} T_{\lambda+k\theta} v_{\Lambda_0} = \epsilon_{s+1} x_{s,s}^-(d_{s,s} + k, d'_{s,s} + \delta_{1,s}k, \underline{\pi(s)}^s) T_{\lambda+k\alpha_{1,s} - \sum_{s < i \leq j \leq r} d_{i,j}\alpha_{i,j} - \sum_{j=s+1}^r d_{s,j}\alpha_{s,j}} f_{s+1} v_{\Lambda_0}, \quad (4.24)$$

where f_{s+1} is a polynomial in $\alpha_i^\vee t^{-j}$, $i \in I, j \in \mathbb{N}$, depends only on the elements of the sets $\mathcal{I}(\mathfrak{P}_{s+1})$ and $\mathcal{I}_s^j(\mathfrak{P})$, $s + 1 \leq j \leq r$, such that the weight of $f_{s+1} v_{\Lambda_0}$ in $L(\Lambda_0)$ is $\Lambda_0 - (d(\mathfrak{P}_{s+1}) + \sum_{j=s+1}^r d_s^j(\mathfrak{P}))\delta$.

Set $\mu = \lambda + k\alpha_{1,s} - \sum_{s < i \leq j \leq r} d_{i,j}\alpha_{i,j} - \sum_{j=s+1}^r d_{s,j}\alpha_{s,j}$. Since

$$\begin{aligned} (\mu|\alpha_{s,s}) &= (\lambda_s^{r+1} - \lambda_{s+1}^{r+1}) + \delta_{1,s}k + k + \sum_{j=s+1}^r d_{s+1,j} - \sum_{j=s}^r d_{s,j} \\ &= (\lambda_s^{r+1} - \lambda_{s+1}^{r+1}) + \delta_{1,s}k + k + (\lambda_{s+1}^{r+1} - \lambda_{s+1}^{s+1}) - (\lambda_s^{r+1} - \lambda_s^{s+1}) \\ &= d_{s,s} + k + d'_{s,s} + \delta_{1,s}k \end{aligned} \quad (4.25)$$

and

$$d_{s,s} \geq d(\mathfrak{P}_s) = |\underline{\pi(s)}^s| + d(\mathfrak{P}_{s+1}) + \sum_{j=s+1}^r d_s^j(\mathfrak{P}),$$

we get the result from (4.24) by using Proposition 4.8 and (2.12).

4.8. We now prove Proposition 3.6. Let π be the representation $L(\Lambda_{i_\lambda})$. Consider the automorphism $r_{w_0}^\pi$ on $L(\Lambda_{i_\lambda})$. It is easy to see that

$$r_{w_0}^\pi(x_\alpha \otimes t^s) r_{w_0}^\pi v = r_{w_0}^{\text{ad}}(x_\alpha \otimes t^s) v = (x_{w_0\alpha} \otimes t^s) v, \quad \forall \alpha \in R, v \in L(\Lambda_{i_\lambda}), s \in \mathbb{Z}_{\geq 0}.$$

Set $v_\lambda := T_{w_0\lambda} v_{\Lambda_0}$. We now observe that

$$r_{w_0}^\pi v_\lambda = T_\lambda v_{\Lambda_0} \quad \text{and} \quad r_{w_0}^\pi v_{\mathfrak{P}} = \epsilon_{\mathfrak{P}} \bar{v}_{\mathfrak{P}}, \quad \forall \mathfrak{P} \in \mathbb{P}_\lambda.$$

Hence the result.

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